Hearn, George W.
Researches on curves
of the second order

Edm. John Senkler
June 11. 1858.

RESEARCHES ON CURVES

OF THE

SECOND ORDER,

ALSO ON

Cones and Spherical Conics treated Analytically,

IN WHICH

THE TANGENCIES OF APOLLONIUS ARE INVESTIGATED,
AND GENERAL GEOMETRICAL CONSTRUCTIONS
DEDUCED FROM ANALYSIS;

ALSO SEVERAL OF

THE GEOMETRICAL CONCLUSIONS OF M. CHASLES
ARE ANALYTICALLY RESOLVED,

TOGETHER WITH

MANY PROPERTIES ENTIRELY ORIGINAL.

BY

GEORGE WHITEHEAD HEARN,

A GRADUATE OF CAMBRIDGE, AND A PROFESSOR OF MATHEMATICS IN THE ROYAL
MILITARY COLLEGE, SANDHURST.

Hearn Jes. 6th Wrangler 1839.
Cory Pemb. 8th ———— ————.

LONDON:
GEORGE BELL, 186, FLEET STREET.
MDCCCXLVI.

6849
27 11 90

LONDON:
PRINTED BY WILLIAM STEVENS, BELL YARD,
TEMPLE BAR.

TABLE OF CONTENTS.

CHAPTER III.

PREFACE.

In this small volume the reader will find no fantastical modes of applying Algebra to Geometry. The old Cartesian or co-ordinate system is the basis of the whole method— and notwithstanding this, the author is satisfied that the reader will find much originality in his performance, and flatters himself that he has done something to amuse, if not to instruct, Mathematicians.

Though the work is not intended as an elementary one, but rather as supplementary to existing treatises on conic sections, any

intelligent student who has digested Euclid, and the usual mode of applying Algebra to Geometry, will meet but little difficulty in the following pages.

Sandhurst,
30*th June,* 1846.

INTRODUCTORY DISCOURSE

CONCERNING

GEOMETRY.

THE ancient Geometry of which the Elements of Euclid may be considered the basis, is undoubtedly a splendid model of severe and accurate reasoning. As a logical system of Geometry, it is perfectly faultless, and has accordingly, since the restoration of letters, been pursued with much avidity by many distinguished mathematicians. Le Père Grandi, Huyghens, the unfortunate Lorenzini, and many Italian authors, were almost exclusively attached to it,—and amongst our English authors we may particularly instance Newton and Halley. Contemporary with these last was the immortal Des Cartes, to whom the analytical or modern system is mainly attributable. That the complete change

of system caused by this innovation was strongly resisted by minds of the highest order is not at all to be wondered at. When men have fully recognized a system to be built upon irrefragable truth, they are extremely slow to admit the claims of any different system proposed for the accomplishment of the same ends; and unless undeniable advantages can be shown to be possessed by the new system, they will for ever adhere to the old.

But the Geometry of Des Cartes has had even more to contend against. Being an instrument of calculation of the most refined description, it requires very considerable skill and long study before the student can become sensible of its immense advantages. Many problems may be solved in admirably concise, clear, and intelligible terms by the ancient geometry, to which, if the algebraic analysis be applied as an instrument of investigation, long and troublesome eliminations are met with,* and the whole solution presents such a contrast to the simplicity of the former method, that a

* This however is usually the fault of the *analyst* and not of the *analysis*.

mind accustomed to the ancient system would be very liable at once to repudiate that of Des Cartes. On the other hand, it cannot be denied that the Cartesian system always presents its results as at once derived from the most elementary principles, and often furnishes short and elegant demonstrations which, according to the ancient method, require long and laborious reasoning and frequent reference to propositions previously established.

It is well known that Newton extensively used algebraical analysis in his geometry, but that, perhaps partly from inclination, and partly from compliance with the prejudice of the times, he translated his work into the language of the ancient geometry.

It has been said, indeed (vide Montucla, part v. liv. I.), that Newton regretted having passed too soon from the elements of Euclid to the analysis of Des Cartes, a circumstance which prevented him from rendering himself sufficiently familiar with the ancient analysis, and thereby introducing into his own writings that form and taste of demonstration which he so much admired in Huyghens and the ancients. Now, much as we may admire the logic

and simplicity of Euclidian demonstration, such has been the progress and so great the achievements of the modern system since the time of Newton, that there seems to be but one reason why we may consider it fortunate that the great "Principia" had previously to seeing the light been translated into the style of the ancients, and that is, that such a style of geometry was the only one then well known. The Cartesian system had at that time to undergo its ordeal, and had the sublime truths taught in the "Principia" been propounded and demonstrated in an almost unknown and certainly unrecognised language, they might have lain dormant for another half century. Newton certainly was attached to the ancient geometry (as who that admires syllogistic reasoning is not?) but he was much too sagacious not to perceive what an instrument of almost unlimited power is to be found in the Cartesian analysis if in the hands of a skilful operator.

The ancient system continued to be cultivated in this country until within very recent years, when the Continental works were introduced by Woodhouse into Cambridge, and it was then soon seen

that in order to keep pace with the age it was absolutely necessary to adopt analysis, without, however, totally discarding Euclid and Newton.

We will now advert to an idea prevalent even amongst analysts, that analytical reasoning applied to geometry is less rigorous or less instructive than geometrical reasoning. Thus, we read in Montucla: "La géométrie ancienne a des avantages qui feroient desirer qu'on ne l'eut pas autant abandonnée. Le passage d'une vérité à l'autre y est toujours clair, et quoique souvent long et laborieux, il laisse dans l'esprit une satisfaction que ne donne point le calcul algébrique qui convainct sans éclairer."

This appears to us to be a great error. That a young student can be sooner taught to comprehend geometrical reasoning than analytical seems natural enough. The former is less abstract, and deals with tangible quantities, presented not merely to the mind, but also to the eye of the student. Every step concerns some line, angle, or circle, visibly exhibited, and the proposition is made to depend on some one or more propositions previously established, and these again on the axioms, postulates,

and definitions; the first being self-evident truths, which cannot be called in question; the second simple mechanical operations, the possibility of which must be taken for granted; and the third concise and accurate descriptions, which no one can misunderstand. All this is very well so far as it goes, and is unquestionably a wholesome and excellent exercise for the mind, more especially that of a beginner. But when we ascend into the higher geometry, or even extend our researches in the lower, it is soon found that the *number of propositions* previously demonstrated, and on which any proposed problem or theorem can be made to depend, becomes extremely great, and that demonstration of the proposed is always the best which combining the requisites of conciseness and elegance, is at the same time the most elementary, or refers to the fewest previously demonstrated or known propositions, and those of the simplest kind. It does not require any very great effort of the mind to remember all the propositions of Euclid, and how each depends on all or many preceding it; but when we come to add the works of Apollonius, Pappus, Archimedes, Huyghens, Halley, Newton, &c.,

that mind which can store away all this knowledge and render it available on the spur of the moment is surely of no common order. Again, the moderns, Euler, Lagrange, D'Alembert, Laplace, Poisson, &c., have so far, by means of analysis, transcended all that the ancients ever did or thought about, that with one who wishes to make himself acquainted with their marvellous achievements it is a matter of imperative necessity that he should abandon the ancient for the modern geometry, or at least consider the former subordinate to the latter. And that at this stage of his proceeding he should by no means form the very false idea that the modern analysis is less rigorous, or less convincing, or less instructive than the ancient syllogistic process. In fact, "more" or "less rigorous" are modes of expression inadmissible in Geometry. If anything is "less rigorous" than "absolutely rigorous" it is no demonstration at all. We will not disguise the fact that it requires considerable patience, zeal, and energy to acquire, *thoroughly understand*, and retain a system of analytical geometry, and very frequently persons deceive themselves by thinking that they fully comprehend an analytical demonstration

when in fact they know very little about it. Nay it is not unfrequent that people write upon the subject who are far from understanding it. The cause of this seems to be, that such persons, when once they have got their proposition translated into equations, think that all they have then to do is to go to work *eliminating* as fast as possible, without ever attempting any *geometrical* interpretation of any of the steps until they arrive at the final result. Far different is the proceeding of those who fully comprehend the matter. To them every step has a geometrical interpretation, the reasoning is complete in all its parts, and it is not the least recommendation of the admirable structure, that it is composed of only a few elementary truths easily remembered, or rather impossible to be forgotten.

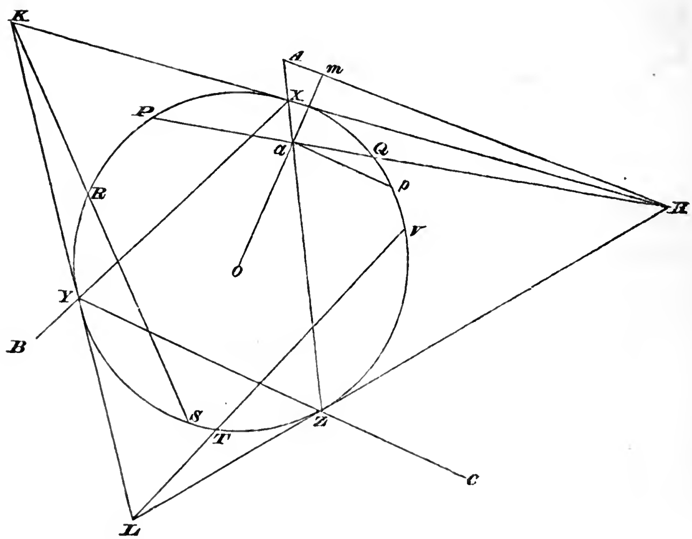
CHAPTER I.

It is intended in this chapter to apply analysis to some problems, which at first view do not seem to be susceptible of concise analytical solutions, and which possess considerable historical interest. The first of these is one proposed by M. Cramer to M. de Castillon, and which may be enunciated thus: "Given three points and a circle, to inscribe in the circle a triangle whose sides shall respectively pass through the given points." See Puissant p. 125.

Concerning this curious problem Montucla remarks that M. de Castillon having mentioned it to Lagrange, then resident at Berlin, this geometer gave him a purely analytical solution of it, and that it is to be found in the Memoirs of the Academy of Berlin (1776), and Montucla then adds, "Elle prouve à la fois la sagacité de son auteur et les ressources de notre analyse, maniée par d'aussi habiles mains." Not having the means of consulting the

Memoir referred to, I have not seen Lagrange's solution, nor indeed any other, and as it has been considered a difficult problem I have considered it a fit subject to introduce into this work as an illustration of the justness of the remarks made in the introductory discourse.

The plan I have adopted is the following :—



Let A B C be the given points. Draw a pair of tangents from A, and let P Q H be the line of contact. Similarly pairs of tangents from B and C, S R K, V T L being lines of contact. Then if a triangle K L H can be described about the circle,

and such that its angular points may be in the given lines P Q H, S R K, V T L respectively, then the points of contact, X, Y, Z being joined will pass respectively through A, B, C. For H being the pole of Z X, tangents drawn where any line H Q P intersects the circle will intersect in Z X produced, but those tangents intersect in A, and therefore Z X passes through A. Similarly of the rest.

When any of the points A, B, C falls within the circle as at a, join $o\ a$. Make $a\ p \perp o\ a\ m$, and draw tangent $p\ m$, then A m H $\perp$ $o\ m$ will hold the place of P Q H in the above.

We have therefore reduced the problem to the following.

Let there be three given straight lines and a given circle, it is required to find a triangle circumscribed about the circle, which shall have its angular points each in one of the three lines.

Let a be the radius of the circle, and let the equations to the required tangents be

$$\left.\begin{array}{l} l_1 x + m_1 y = a \\ l_2 x + m_2 y = a \\ l_3 x + m_3 y = a \end{array}\right\} (1)$$

Also the equations to the three given lines

$$\left.\begin{aligned} A_1 x + B_1 y &= p_1 \\ A_2 x + B_2 y &= p_2 \\ A_3 x + B_3 y &= p_3 \end{aligned}\right\} (2)$$

p_1, p_2, p_3 being perpendiculars upon them from the centre of the circle, and l_1, m_1, A_1, B_1, &c., direction cosines.

Suppose the intersection of the two first lines of (1) to be in the third line of (2), we have by eliminating x and y between

$$l_1 x + m_1 y = a$$
$$l_2 x + m_2 y = a$$
$$\text{and } A_3 x + B_3 y = p_3$$

the condition

$$A_3 a (m_2 - m_1) + B_3 a (l_1 - l_2) = p_3 (l_1 m_2 - l_2 m_1)$$

and similarly

$$A_2 a (m_1 - m_3) + B_2 a (l_3 - l_1) = p_2 (l_3 m_1 - l_1 m_3)$$
$$A_1 a (m_3 - m_2) + B_1 a (l_2 - l_3) = p_1 (l_2 m_3 - l_3 m_2)$$

Now let $l_1 = \cos \theta_1$, $\therefore m_1 = \sin \theta_1$ &c.

Also $A_3 = \cos \alpha_3$, $\therefore B_3 = \sin \alpha_3$ &c.

Then the first of the above conditions is

$$a\{\cos \alpha_3 (\sin \theta_2 - \sin \theta_1) + \sin \alpha_3 (\cos \theta_1 - \cos \theta_2)\}$$
$$= p_3 \sin (\theta_2 - \theta_1)$$

This equation is easily reducible by ordinary trigonometry to

$$\tan\frac{\alpha_3 - \theta_1}{2}\tan\frac{\alpha_3 - \theta_2}{2} + \frac{p_3 - a}{p_3 + a} = 0$$

Similarly

$$\tan\frac{\alpha_2 - \theta_3}{2}\tan\frac{\alpha_2 - \theta_1}{2} + \frac{p_2 - a}{p_2 + a} = 0$$

$$\tan\frac{\alpha_1 - \theta_2}{2}\tan\frac{\alpha_1 - \theta_3}{2} + \frac{p_1 - a}{p_1 + a} = 0$$

If now for brevity we put $x = \tan\frac{\theta_1}{2}$, $y = \tan\frac{\theta_2}{2}$, $z = \tan\frac{\theta_3}{2}$, also $k_3 = \frac{p_3 + a\cos\alpha_3}{p_3 - a\cos\alpha_3}$, $h_3 = \frac{a\sin\alpha_3}{p_3 - a\cos\alpha_3}$ &c. the above equations become

$$k_3\,x\,y - h_3\,(x + y) + 1 = 0$$

$$k_2\,z\,x - h_2\,(z + x) + 1 = 0$$

$$k_1\,y\,z - h_1\,(y + z) + 1 = 0$$

from which we can immediately deduce a quadratic for x.

On eliminating z between the second and third equations, we shall have another equation in x and y similar in form to the first.

We may, moreover, so assume the axis from which

a_1, a_2, a_3 are measured, so that $h_3 = 0$ and the equations are then,

$$k_3\, x\, y + 1 = 0$$

$$\text{and } (h_2\, k_1 - h_1\, k_2)\, x\, y + (h_1\, h_2 - k_1)\, y -$$
$$(h_1\, h_2 - k_2)\, x + h_{\cancel{2}\,1} - h_{\cancel{1}\,2} = 0.$$

These are, considering x and y as co-ordinates, the equations to two hyperbolas having parallel asymptotes, and which we may assume to be rectangular. To show that their intersections may be easily determined geometrically, assume the equations under the form

$$x\, y = C^2$$

$$x\, y - C^2 + \mu \left(\frac{x}{A} + \frac{y}{B} - 1 \right) = 0$$

Then by subtraction,

$$\frac{x}{A} + \frac{y}{B} - 1 = 0$$

is the common secant.

Let
$$\frac{x}{B} + \frac{y}{A} - 1 = 0$$

be another secant.

Multiply these together and we have

$$\frac{x^2 + y^2}{A\, B} + \frac{A^2 + B^2}{A^2\, B^2}\, x\, y - \frac{A + B}{A\, B}\, (x + y) + 1 = 0$$

This equation represents the *two* secants. But at the points of their intersection with the hyperbola $x\,y = C^2$, this last equation reduces to

$$x^2 + y^2 - (A + B)\,(x + y) + A\,B + C^2\frac{A^2 + B^2}{A\,B} = 0$$

$$\text{or } \left(x - \frac{A + B}{2}\right)^2 + \left(y - \frac{A + B}{2}\right)^2$$

$$= (A^2 + B^2)\left\{\frac{A\,B - 2C^2}{2\,A\,B}\right\}$$

which represents a circle, co-ordinates of the centre

$$x = y = \frac{A + B}{2}$$

$$\text{and radius } (A^2 + B^2)^{\frac{1}{2}}\left\{\frac{A\,B - 2C^2}{2\,A\,B}\right\}^{\frac{1}{2}}$$

Hence it is evident that A and B, being once geometrically assigned, the rest of the construction is merely to draw this circle, which will intersect $\frac{x}{A} + \frac{y}{B} - 1 = 0$ in the required points.

The analytical values of A, B, and C^2 are

$$A = -\frac{(h_{\cancel{1}\,2} - h_{\cancel{2}\,1})\,k_3 + h_2\,k_1 - h_1\,k_2}{(h_1\,h_2 - k_2)\,k_3} \qquad C^2 = -\frac{1}{k_3}$$

$$B = \frac{(h_{\cancel{1}\,2} - h_{\cancel{2}\,1})\,k_3 + h_2\,k_1 - h_1\,k_2}{(h_1\,h_2 - k_1)\,k_3}.$$

These being rational functions of known geo-

metrical magnitudes, are of course assignable *geometrically*, so that every difficulty is removed, and the mere labour of the work remains.

In the next place, I propose to derive a general mode of construction for the various cases of the "tangencies" of Apollonius from analysis. The general problem may be stated thus: of three points, three lines and three circles, any three whatever being given, to describe another circle touching the given lines and circles and passing through the given points.

It is very evident that all the particular cases are included in this, "to describe a circle touching three given circles," because when the centre of a circle is removed to an infinite distance, and its radius is also infinite, that circle becomes at all finite distances from the origin a straight line. Also, when the radius of a circle is zero it is reduced to a point.

We will therefore proceed at once to the consideration of this problem, and it is hoped that the construction here given will be found more simple than any hitherto devised.

The method consists in the application of the two following propositions.

If two conic sections have the same focus, lines may be drawn through the point of intersection of their *citerior* directrices,* and through two of the points of intersection of the curves.

Let u and v be linear functions of x and y, so that the equations $u = o$, $v = o$ may represent the citerior directrices, then if $r = \sqrt{x^2 + y^2}$, and m and n be constants, we have for the equations of the two curves

$$r = m\,u$$
$$r = n\,v$$

and by eliminating r, $m\,u - n\,v = o$; but this is the equation to a straight line through the intersection of $u = o$, $v = o$, since it is satisfied by these simultaneous equations.

When the curves are both ellipses they can intersect only in two points, and the above investigation is fully sufficient. But when one or both the curves are hyperbolic, we must recollect that only one branch of each curve is represented by each of the above equations. The other branches are,

$$r = -\,m\,u$$
$$r = -\,n\,v$$

* By the term "citerior" I mean those directrices nearest to the common focus.

We have, therefore, in this instance $m\,u + n\,v = 0$ as well as $m\,u - n\,v = 0$, for a line of intersection.

The second proposition is, having given the focus, citerior directrix, and eccentricity of a conic section, to find by geometrical construction the two points in which the conic section intersects a given straight line.

In either of the diagrams, the first of which is for an ellipse, the second for a hyperbola, let M X be the given straight line, F the focus, A the vertex, and D R the citerior directrix. Let

$$\text{F M} \perp \text{M X} = p, \ \text{M F D} = \alpha,$$

r the distance of any point in M X from F, θ the angle it makes with F D, and F D $= a$.

Also let $n = \dfrac{\text{F A}}{\text{A D}}$

Then $\dfrac{r}{a - r\cos\theta} = n, \qquad r\cos(\theta - \alpha) = p$

Eliminate r

$$\frac{a}{p}\cos(\theta - \alpha) - \cos\theta = \frac{1}{n}$$

or $$(a\cos\alpha - p)\cos\theta + a\sin\alpha\sin\theta = \frac{p}{n}$$

Let $$\frac{a \cos \alpha - p}{a \sin \alpha} = \cot \varepsilon \quad . \quad . \quad . \quad . \quad . \quad (1)$$

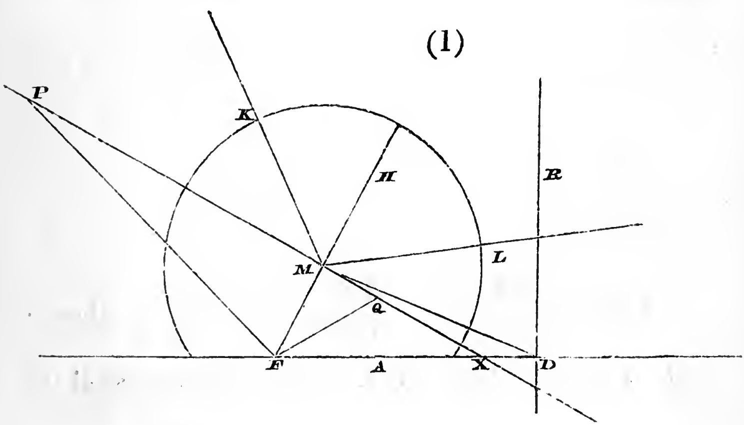
Then $$\frac{a \sin \alpha}{\sin \varepsilon} \cos (\theta \sim \varepsilon) = \frac{p}{n}$$

or $$\cos (\theta \sim \varepsilon) = \frac{p \sin \varepsilon}{n\, a \sin \alpha} = \frac{p}{n\, d} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

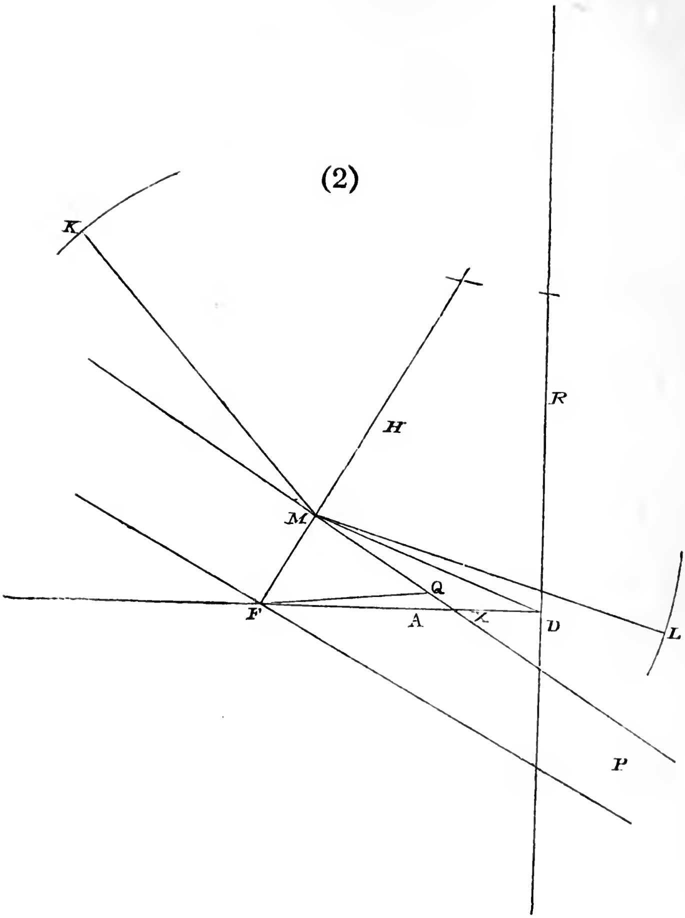
where $d = \dfrac{a \sin \alpha}{\sin \varepsilon}$

From the formulæ (1) and (2) we derive the following construction. Join D M, then D M H is the angle ε, because D M projected on F H is $a \cos \alpha - p$. Also a perpendicular from D on F H is $a \sin \alpha$, $\therefore \dfrac{a \cos \alpha - p}{a \sin \alpha} = \cot$ D M H, $\therefore \cot$ D M H $= \cot \varepsilon$, $\therefore$ D M H $= \varepsilon$

(1)



(2)



Again, $\frac{\sin D M F}{\sin D F M} = \frac{D F}{D M}$, or $\frac{\sin \epsilon}{\sin \alpha} = \frac{a}{D M}$, whence $D M = d$. Find $M L$ a third proportional to

A D, F A, and D M, so that $M L = n d$. With centre M and radius M L describe a circle. Make M H equal to F M, and draw K H L at right angles to F H, and join M K, M L. Then by (2) L M H or K M H $= \theta \sim \varepsilon$.

Taking the value $\varepsilon - \theta$, we have therefore L M D $=$ D M H $-$ L M H $= \varepsilon - (\varepsilon - \theta) = \theta$.
And taking $\theta - \varepsilon$, we have K M D $=$ K M H $+$ H M D $= \varepsilon + \theta - \varepsilon = \theta$.

Hence, make Q F X $=$ L M D, P F D $=$ K M D, and P and Q are the two points required.

We now proceed to show how, by combining these two propositions, the circles capable of simultaneously touching three given circles may be found.

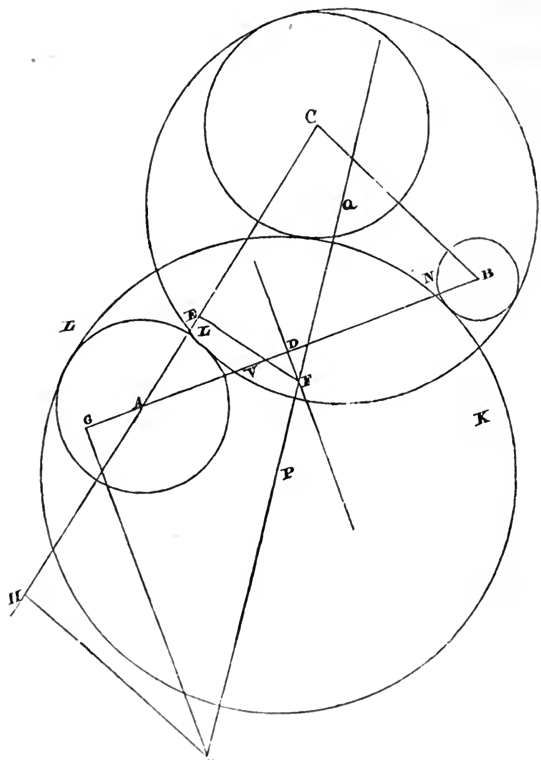
Let A, B, C, be the centres of the three circles, and let the sides of the triangle A B C be as usual denoted by a, b, c; the radii of the circles being α, β, γ.

We will suppose that the circle required *envelopes* A and touches B and C externally, and the same process, *mutatis mutandis*, will give the other circles.

Taking A B for axis of x, and A for origin, we easily find in the usual way the equation to the

hyperbola, which is the locus of the centres of the circles touching A and B.

$$r = \frac{c^2 - (\alpha + \beta)^2 - 2\,c\,x}{2\,(\alpha + \beta)}$$



From which, D F being the citerior directrix, we have $A\,D = \dfrac{c^2 - (\alpha + \beta)^2}{2\,c}$ Hence, with radius

B K = $\alpha + \beta$ describe an arc. Bisect A B, and from its middle point as centre and rad. $\frac{1}{2}$ A B describe an arc, intersecting the former in K. Draw K N $\perp$ A B, and bisect A N in D, then D F $\perp$ A B is the citerior directrix. Again, make A V to A D as c to $\alpha + \beta + c$, *i. e.* as A B to rad. A + rad. B + A B, and V will be the citerior vertex.

Assign the citerior directrix E F of the hyperbola, which is the locus of the circles touching A and C. Make D G to E H in the ratio compounded of the ratios of b to c, and $\alpha + \beta$ to $\alpha + \gamma$. Draw G S and H S $\perp$ to B A and C A, and through S and F draw S P F Q; this will be the line of centres, and by applying the second proposition, two points, P and Q, will be found. Join P A, and produce it to meet the circle A in L, and with radius P L describe a circle, and this will envelope A and touch B and C externally. Also, if Q A be joined, cutting circle A in L^1, and a circle radius Q L^1 be described, it will envelope B and C, and touch A externally.

Similarly the three other pairs of circles may be found.

As it would too much increase the extent of this work to go *seriatim* through the several cases of the tangencies—that is, to apply the foregoing propositions to each case, the reader is supposed to apply them himself.

I have in the "Mathematician," vol. 1, p. 228, proposed and proved a curious relation amongst the radii of the eight tangent circles. The following is another curious property.

With reference to the last figure, suppose we denote the hyperbolic branch of the locus of the centres of circles enveloping A and touching B externally by $A_c\,B_u$, $A_u\,B_c$, the former meaning "branch citerior to A and ulterior to B," the latter "citerior to B and ulterior to A." The six hyperbolic branches will then be thus denoted :

$$A_c\,B_u,\ A_u\,B_c;\ B_c\,C_u,\ B_u\,C_c;\ C_c\,A_u,\ C_u\,A_c$$

and suppose the corresponding directrices denoted thus :

$$\overline{A_c\,B_u},\ \overline{A_u\,B_c};\ \overline{B_c\,C_u},\ \overline{B_u\,C_c};\ \overline{C_c\,A_u},\ \overline{C_u\,A_c}$$

Then the point P is the mutual intersection of

$$A_c\,C_u,\ A_c\,B_u,\ B_c\,C_u$$

and Q is the mutual intersection of

$$A_u\,C_c,\ A_u\,B_c,\ B_u\,C_c$$

P Q passes through intersection of $\overline{A_c\,B_u,\ A_c\,C_u}$ because it passes through intersections of $A_c\ B_u$, $A_c\ C_u$, and of $A_u\ C_c$, $A_u\ B_c$.

Also P Q through $\overline{B_c\,A_u,\ B_c\,C_u}$, because through $B_c\ A_u$, $B_u\ C_c$ and $B_u\ A_c$, $B_c\ C_u$.

Also P Q through $\overline{C_c\,A_u,\ C_c\,B_u}$, because through $C_c\ A_u$, $C_c\ B_u$ and $C_u\ A_c$, $C_u\ B_c$, and hence the intersections

$\overline{A_c\,B_u,\ A_c\,C_u}$; $\overline{B_c\,A_u,\ B_c\,C_u}$; $\overline{C_c\,A_u,\ C_c\,B_u}$ are all in the same straight line P Q.

That is, the intersections of pairs of directrices citerior respectively to A, B, C are in the same straight line, namely, the line of centres of the pair of tangent circles to which they belong.

CHAPTER II.

On curves of the second order passing through given points and touching given straight lines.

Let $u = 0$, $v = 0$, $w = 0$, be the equations to three given straight lines.

The equation

$$\lambda\, v w + \mu\, u w + \nu\, u v = 0 \quad . \; . \; . \quad (1)$$

being of the second order represents a conic section, and since this equation is satisfied by any two of the three equations $u = 0$, $v = 0$, $w = 0$, (1) will pass through the three points formed by the mutual intersections of those lines.

To assign values of λ, μ, ν, in terms of the co-ordinates of the centre of (1),

We have

$$u = a_2 x + b_2 y + 1$$
$$v = a_3 x + b_3 y + 1$$
$$w = a_4 x + b_4 y + 1$$

Hence (1) differentiated relatively to x and y will give

$$\left.\begin{array}{l}\lambda\ \{a_4 v+a_3 w\}+\mu\{a_2 w+a_4 u\}+\nu\{a_3 u+a_2 v\}=0\\ \lambda\ \{b_4 v+b_3 w\}+\mu\{b_2 w+b_4 u\}+\nu\{b_3 u+b_2 v\}=0\end{array}\right\}\ldots(a)$$

and these are the equations for finding the co-ordinates of the centre.

Let now L, M, and N be three such quantities that

$$\mathrm{L}\,u+\mathrm{M}\,v+\mathrm{N}\,w$$

may be *identically* equal to 2 K, then by finding the ratios $\frac{\lambda}{\mu}$, $\frac{\lambda}{\nu}$ from (a) it will be found that the following values may be assigned to λ, μ, ν,

$$\lambda = u\,(\mathrm{L}\,u - \mathrm{K}),\ \mu = v\,(\mathrm{M}\,v - \mathrm{K}),\ \nu = w\,(\mathrm{N}\,w - \mathrm{K})$$

Hence when any relation exists amongst λ, μ, ν, we can, by the substitution of these values, immediately determine the locus of the centres of (1).

1° Let (1) pass through a fourth point, then λ, μ, ν, are connected by the relation

$$\mathrm{A}\,\lambda + \mathrm{B}\,\mu + \mathrm{C}\,\nu = 0 \quad \ldots \quad (a)$$

where A, B, C are the values of $v\,w$, $u\,w$, $u\,v$, for the fourth point.

Hence the locus of the centres of all conic sections drawn through the four points will be

$$\mathrm{A}\,u\,(\mathrm{L}\,u - \mathrm{K}) + \mathrm{B}\,v\,(\mathrm{M}\,v - \mathrm{K}) + \mathrm{C}\,w\,(\mathrm{N}\,w - \mathrm{K}) = 0 \quad \ldots \quad (b)$$

which is itself a curve of the second order.

2° When the fourth point coincides with one of the other points, the values of A, B, C vanish. But suppose the fourth point infinitely near to the intersection of $u = 0, v = 0$, and that it lies in the straight line $u + n\,v = 0$. Then since on putting $x+h, y+k$ for x and y, we have

$$(v\,w)^1 = (v\,w) + (a_4\,v + a_3\,w)\,h + (b_4\,v + b_3\,w)\,k$$

and ∴
$$\mathrm{A} = \mathrm{w}\,(a_3\,h + b_3\,k) \qquad \mathrm{C} = 0$$
$$\mathrm{B} = \mathrm{w}\,(a_2\,h + b_2\,k)$$

Where w is the value of w, for the values of x and y determined by $u = 0 \;\; v = 0$

Moreover from the equation $u + n\,v = 0$

$$a_2\,h + b_2\,k + n\,(a_3\,h + b_3\,k) = 0$$

Hence $$\mathrm{B} + n\,\mathrm{A} = 0$$

and $$\mathrm{A}\,\lambda + \mu\,\mathrm{B} = 0$$

and ∴ $$\lambda - n\,\mu = 0$$

∴ $$u\,(\mathrm{L}\,u - \mathrm{K}) - n\,v\,(\mathrm{M}\,v - \mathrm{K}) = 0$$

is the ultimate state of equation (b). This latter is therefore the locus of the centres of all conic sections which can be drawn through two given points $u\,w$, $v\,w$, and touching a given straight line $u + n\,v = 0$ in a given point $u\,v$.

3° Let λ, μ, ν be connected by the equation

$$(\mathrm{A}\,\lambda)^{\frac{1}{2}} + (\mathrm{B}\,\mu)^{\frac{1}{2}} + (\mathrm{C}\,\nu)^{\frac{1}{2}} = 0 \ldots (\beta)$$

and in conformity with this condition let us seek the *envelope* of (1):

Diff. (1) and (β) relatively to λ, μ, ν, we have

$$v\,w\,d\,\lambda + u\,w\,d\,\mu + u\,v\,d\,\nu = 0$$

or

$$\frac{1}{u}\,d\,\lambda + \frac{1}{v}\,d\,\mu + \frac{1}{w}\,d\,\nu = 0$$

$$\frac{A^{\frac{1}{2}}}{\lambda^{\frac{1}{2}}}\,d\,\lambda + \frac{B^{\frac{1}{2}}}{\mu^{\frac{1}{2}}}\,d\,\mu + \frac{C^{\frac{1}{2}}}{\nu^{\frac{1}{2}}}\,d\,\nu = 0$$

Hence $\lambda^{\frac{1}{2}} = k\,A^{\frac{1}{2}}\,u$, $\mu^{\frac{1}{2}} = k\,B^{\frac{1}{2}}\,v$, $\nu^{\frac{1}{2}} = k\,B^{\frac{1}{2}}\,w$, putting which in ($\beta$) we have

$$A\,u + B\,v + C\,w = 0$$

for the envelope required. We may therefore consider (β) as the condition that the curve (1) passing through three given points may also touch a given straight line $t = 0$, for we have only to determine A, B, and C, so that

$$A\,u + B\,v + C\,w = t$$

identically. Substituting the values of λ, μ, ν, in (β) we have for the locus of the centres of a system of conic sections passing through three given points and touching a given straight line,

$$\left\{A\,u\,(L\,u - K)\right\}^{\frac{1}{2}} + \left\{B\,v\,(M\,v - K)\right\}^{\frac{1}{2}} + \left\{C\,w\,(N\,w - K)\right\}^{\frac{1}{2}} = 0 \quad . \; . \; . \quad (c)$$

which being rationalized will be found to be of the fourth order.

4° Let $u = 0$, $v = 0$, $w = 0$ be the equations to three given straight lines,

$$(\lambda u)^{\frac{1}{2}} + (\mu v)^{\frac{1}{2}} + (\nu w)^{\frac{1}{2}} = 0 \quad . \ . \ . \ (2)$$

will be the equation necessary to a conic section touching each of those lines.

For the equation in a rational form is

$$\lambda^2 u^2 + \mu^2 v^2 + \nu^2 w^2 = 2 \{\lambda \mu u v + \lambda \nu u w + \mu \nu v w\}$$

Make $w = 0$ and it reduces to

$$(\lambda u - \mu v)^2 = 0$$

and hence the points common to (2) and $w = 0$ will be determined by the simultaneous equations $w = 0$ and $\lambda u - \mu v = 0$. But these being linear, determine only one point. Hence $w = 0$ is a tangent to (5). Similarly $u = 0$, $v = 0$ are tangents.

5° Let λ, μ, ν, be connected by the equation

$$\frac{\lambda}{A} + \frac{\mu}{B} + \frac{\nu}{C} = 0 \quad . \ . \ . \ (\gamma)$$

where A, B, C are fixed constants, and consistently with this condition let us seek the envelope of (2).

Differentiating (2) and (γ) with respect to λ, μ, ν,

$$\lambda^{-\frac{1}{2}} u^{\frac{1}{2}} d \lambda + \mu^{-\frac{1}{2}} v^{\frac{1}{2}} d \mu + \nu^{-\frac{1}{2}} w^{\frac{1}{2}} \partial \nu = 0$$

$$\frac{d\lambda}{A} + \frac{d\mu}{B} + \frac{d\nu}{C} = 0$$

Hence $\quad \dfrac{k}{A} = \lambda^{-\frac{1}{2}} u^{\frac{1}{2}}$ or $k^2 \lambda = A^2 u$

k being an arbitrary factor.

Also $\quad k^2 \mu = B^2 v \qquad k^2 \nu = C^2 w$

putting which in (2) we have

$$A u + B v + C w = 0$$

for the envelope required, and which being linear represents a straight line.

Hence, if $t = 0$ be the equation to a fourth straight line, and A, B, C be determined by making

$$A u + B v + C w \text{ identical with } t$$

equation (2) subject to the condition (γ) will represent all conic sections capable of simultaneously touching four given straight lines,

$$t = 0, \quad u = 0, \quad v = 0, \quad w = 0.$$

Expanding the equation (2) into its rational integral form, and differentiating with respect to x and y, and putting the differential co-efficients $\dfrac{d\,(2)}{d\,x}, \dfrac{d\,(2)}{d\,y}$ separately $= 0$, we get two equations

for the co-ordinates of the centre. Those equations may be exhibited thus:

$$\frac{\lambda\,(a_2\,b_4 - a_4\,b_2) + \mu\,(a_4\,b_3 - a_3\,b_4)}{w}$$

$$= \frac{\nu\,(a_4\,b_3 - a_3\,b_4) + \lambda\,(a_3\,b_2 - a_2\,b_3)}{v}$$

$$= \frac{\mu\,(a_3\,b_2 - a_2\,b_3) + \nu\,(a_2\,b_4 - a_4\,b_2)}{u}$$

or, $$\frac{\lambda\,\mathrm{M} + \mu\,\mathrm{L}}{w} = \frac{\nu\,\mathrm{L} + \lambda\,\mathrm{N}}{v} = \frac{\mu\,\mathrm{N} + \nu\,\mathrm{M}}{u}$$

where L, M, N are determined as before by making

$$\mathrm{L}\,u + \mathrm{M}\,v + \mathrm{N}\,w = 2\,\mathrm{K} \text{ identically.}$$

The preceding equations give

$$\lambda = \mathrm{L}\,(\mathrm{L}\,u - \mathrm{K}) \qquad \mu = \mathrm{M}\,(\mathrm{M}\,v - \mathrm{K})$$

$$\nu = \mathrm{N}\,(\mathrm{N}\,v - \mathrm{K})$$

and putting these in the condition (γ) we have

$$0 = \frac{\mathrm{L}\,(\mathrm{L}u - \mathrm{K})}{\mathrm{A}} + \frac{\mathrm{M}\,(\mathrm{M}\,v - \mathrm{K})}{\mathrm{B}} + \frac{\mathrm{N}\,(\mathrm{N}w - \mathrm{K})}{\mathrm{C}}$$

for the locus of centres which being linear in u, v, w, will be linear in x and y, and therefore represents a straight line.

6°. Resuming again the equation (2), and making λ, μ, ν, subject to the condition

$$(\mathrm{A}\,\lambda)^{\frac{1}{2}} + (\mathrm{B}\,\mu)^{\frac{1}{2}} + (\mathrm{C}\,\nu)^{\frac{1}{2}} = 0,$$

which will restrict the curve (2) to pass through a given point, A, B, C being the values of u, v, and w, for that point. Putting in the values of λ, μ, ν, determined above, we have

$$\{A\,L\,(L\,u - K)\}^{\frac{1}{2}} + \{B\,M\,(M\,v - K)\}^{\frac{1}{2}} + \{C\,N\,(N\,w - K)\}^{\frac{1}{2}} = 0$$

for the locus of centres.

Hence the locus of the centres of all conic sections which touch three given straight lines and pass through a given point is also a conic section.

Cor. From the form of the equation this locus touches the lines $u = \frac{K}{L}$, $v = \frac{K}{M}$, $w = \frac{K}{N}$, which are parallel to the given lines and at the same distances from them respectively wherever the *given* point may be situated, L, M, N, K, being independent of A, B, C. In fact, it is easy to demonstrate that they are the three straight lines joining the points of bisection of the sides of the triangle formed by u, v, w, and hence the following theorem.

If a system of conic sections be described to pass through a given point and to touch the sides of a given triangle, the locus of their centres will be another conic section touching the sides of the co-polar triangle which is formed by the lines join-

ing the points of bisection of the sides of the former.

7°. We now proceed to the case of a conic section touching two given straight lines, and passing through two given points. Let $u = 0$, $v = 0$, be the equations of the two lines touched, and $w = 0$ the equation of the line passing through the two given points. Then taking the equation

$$(\lambda\, u)^{\frac{1}{2}} + (\mu\, v)^{\frac{1}{2}} + (\nu\, w + 1)^{\frac{1}{2}} = 0$$

we know by the preceding that this represents a conic section touching $u = 0$, $v = 0$, and $w + \frac{1}{\nu} = 0$.

Let α, α' be the values of u at the given points, and β, β' those of v, the values of w being zero for each, then the equations for finding λ and μ will be

$$(\lambda\, \alpha)^{\frac{1}{2}} + (\mu\, \beta)^{\frac{1}{2}} + 1 = 0$$
$$(\lambda\, \alpha')^{\frac{1}{2}} + (\mu\, \beta')^{\frac{1}{2}} + 1 = 0\,;$$

Let A and B be the values of λ and μ deduced from these, and we have for the equation of the conic section

$$(\mathrm{A}\, u)^{\frac{1}{2}} + (\mathrm{B}\, v)^{\frac{1}{2}} + (\nu\, w + 1)^{\frac{1}{2}} = 0,$$

in which ν is the only arbitrary constant.

Differentiating this equation when expanded into

its rational form with respect to x and y, we have two equations respectively equivalent to

$$\nu = - \mathrm{N} \cdot \frac{\mathrm{A}\,u - \mathrm{B}\,v}{\mathrm{L}\,u - \mathrm{M}\,v}$$

$$\nu w + 1 = \frac{\mathrm{A\,M} + \mathrm{B\,L}}{\mathrm{A\,M} - \mathrm{B\,L}} (\mathrm{A}\,u - \mathrm{B}\,v)$$

determining L, M, N, as before, by making

$$\mathrm{L}\,u + \mathrm{M}\,v + \mathrm{N}\,w = 2\,\mathrm{K} \text{ identically.}$$

Hence eliminating ν, there arises

$$2\,\{\mathrm{B\,L}\,(\mathrm{L}\,u - \mathrm{K}) - \mathrm{A\,M}\,(\mathrm{M}\,v - \mathrm{K})\}\,(\mathrm{A}\,u - \mathrm{B}\,v) = (\mathrm{A\,M} - \mathrm{B\,L})\,(\mathrm{L}\,u - \mathrm{M}\,v)$$

for the locus of the centres.

This is also a curve of the second order, and the values of A and B are

$$\mathrm{A} = \left\{\frac{\beta^{\frac{1}{2}} - \beta'^{\frac{1}{2}}}{(\alpha\,\beta')^{\frac{1}{2}} - (\alpha'\,\beta)^{\frac{1}{2}}}\right\}^2$$

$$\mathrm{B} = \left\{\frac{\alpha'^{\frac{1}{2}} - \alpha^{\frac{1}{2}}}{(\alpha\,\beta')^{\frac{1}{2}} - (\alpha'\,\beta)^{\frac{1}{2}}}\right\}^2$$

This demonstration assumes that it is possible to draw a tangent to each of the system of curves parallel to $w = 0$. But in case the given points are in opposite vertical angles of the given straight lines, and the curves therefore hyperbolas, this will not be possible, and accordingly in such case the

values of A and B become imaginary, for in this case α, α', as also β, β' have different signs. The following method is free from this and every objection, and is perfectly general.

8°. Let $u = 0$, and $v = 0$, as before, be the equations to the tangents, $w = 0$ the straight line joining the two given points, and $w' = a'_4\, x + b'_4\, y$, a'_4 and b'_4 being determined as follows:

$$w'_{\alpha\,\beta} = a'_4\, \alpha + b'_4\, \beta = (u\,v)^{\frac{1}{2}}_{\alpha\,\beta}$$

$$w'_{\alpha'\,\beta'} = a'_4\, \alpha' + b'_4\, \beta' = (u\,v)^{\frac{1}{2}}_{\alpha'\,\beta'}$$

$\alpha\,\beta$; $\alpha'\,\beta'$ being co-ordinates of the given points. Then

$$(w + m\,w')^2 = m^2\,u\,v$$

is the equation to the system in which m is arbitrary. For $u = 0$, or $v = 0$, each give $w + m\,w' = 0$, and therefore u and v each touch the curve, and $w + m\,w' = 0$ is the equation to the line joining their points of contact. Again, by the preceding determination of w', we have for $x = \alpha$, $y = \beta$, $w = 0$, and $m^2\, w'^2_{\alpha\,\beta} = m^2\,(u\,v)_{\alpha\,\beta}$ and similar for $\alpha'\,\beta'$, and hence m remains arbitrary. Differentiating the equation

$$(w + m\,w')^2 = m^2\,u\,v$$

with respect to x and y,

$$m^2 u v = (w + m w')^2$$

$$m^2 (a_2 v + a_3 u) = 2 (w + m w') (a_4 + m a'_4)$$

$$m^2 (b_2 v + b_3 u) = 2 (w + m w') (b_4 + m b'_4)$$

$$m^2 \{L u - M v\} = 2 m (w + m w') (a'_4 b_4 - a_4 b'_4)$$

$$m \{L u - M v\} = 2 Q (w + m w')$$

$$m^2 \{L' u - M' v\} = - 2 Q (w + m w')$$

$$\therefore m = - \frac{L u - M v}{L' u - M' v}$$

$$\therefore w + m w' = w - \frac{L u - M v}{L' u - M' v} w'$$

$$\frac{(L u - M v)^2}{L' u - M' v} + 2 Q \left\{ w - \frac{L u - M v}{L' u - M' v} w' \right\} = 0 ;$$

or,

$$(Lu - Mv)^2 + 2 Q\{(L'u - M'v) w - (L u - M v) w'\} = 0,$$

which is an equation of the second order.

Now let $u_{\alpha \beta}$, $v_{\alpha \beta}$ be both positive, and $u_{\alpha' \beta'}$, $v_{\alpha' \beta'}$ both negative, and therefore the given points in opposite vertical angles of the straight lines $u = 0$ and $v = 0$. Then a'_4 and b'_4 will both be real quantities, and $\therefore$ also Q, L′, M′, and w'. Also if $u_{\alpha \beta}$, $v_{\alpha \beta}$ have different signs, as also $u_{\alpha' \beta'}$, $v_{\alpha' \beta'}$, then

a'_4, b'_4, Q, L′, M′, and w' will be of the form $A\sqrt{-1}$, and the above equation equally real.

9°. We have now discussed the several cases of the general problem, whose enunciation is as follows:

Of four straight lines and four points let any four be given, and draw a system of conic sections passing through the given points, and touching the given lines, to investigate the locus of the centres.

We have shown that in every case except two the locus is a conic section. The two exceptions are, first, when there are three given points and a given straight line, in which case the locus is

$$\{A\,u\,(L\,u-K)\}^{\frac{1}{2}}+\{B\,v\,(M\,v-K)\}^{\frac{1}{2}}$$
$$+\{C\,w\,(N\,w-K)\}^{\frac{1}{2}}=0,$$

which, being rationalized, is of the fourth order.

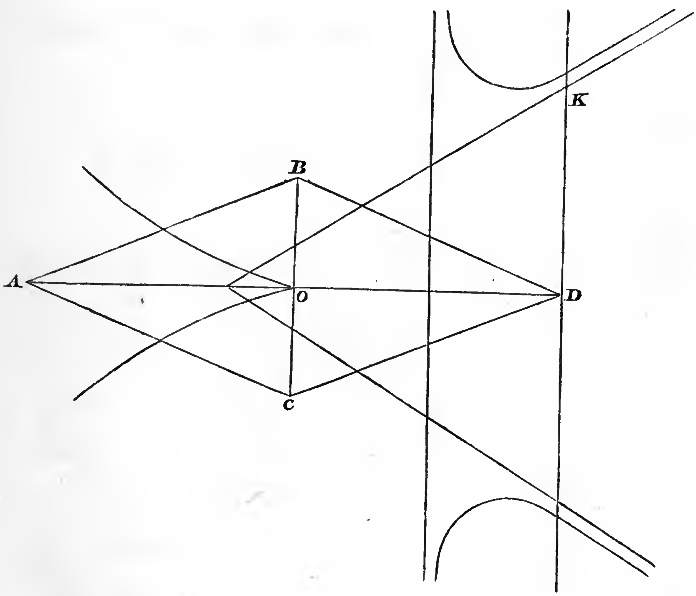
But some doubt may exist as to whether such equation may not be decomposable into two quadratic factors, and thus represent two conic sections. That such cannot hold generally will best appear from the discussion of a particular case.

The other case of exception is when the data are four straight lines, the locus then being a straight line; but since a straight line may be included

amongst the conic sections, we may say that there is but one case of exception.

The particular case we propose to investigate is the following.

Through one of the angular points of a rhombus draw a straight line parallel to a diagonal, and let a system of conic sections be drawn, each touching the parallel to the diagonal, and also passing through the three remaining angular points of the rhombus, to investigate the locus of centres.



Let AO $=$ O D $=$ 1, tan B A O $=$ tan C A O $= m$.

Taking the origin at O the equations are

$$u = y - m\,(x + 1) = 0 \text{ for A B}$$

$$v = y + m\,(x + 1) = 0 \text{ for A C}$$

$$w = x = 0 \qquad\qquad \text{for B C}$$

$$A\,u + B\,v + C\,w = x - 1 = 0 \text{ for K D.}$$

To determine A, B, C, we have therefore

$$A\left(y - m\,(x + 1)\right) + B\left(y + m\,(x + 1)\right) + C\,x$$

$$= x - 1$$

identically ; $\qquad \therefore A + B = 0$

$$- m\,A + m\,B + C = 1$$

$$- m\,A + m\,B = - 1$$

$$\therefore C = 2, \quad A = \frac{1}{2\,m}, \quad B = - \frac{1}{2\,m}.$$

Also for finding L, M, N

$$L\left(y - m\,(x + 1)\right) + M\left(y + m\,(x + 1)\right) + N\,x = 2\,K$$

$$L + M = 0$$

$$- m\,L + m\,M + N = 0$$

$$- m\,L + m\,M = 2\,K$$

$$\therefore N = - 2K, L = - \frac{K}{m}, M = \frac{K}{m}$$

by the substitution of which in (c) we have for the locus of the centres,

$$\{u(u+m)\}^{\frac{1}{2}} + \{v(v-m)\}^{\frac{1}{2}} + \{4m^2 w(2w+1)\}^{\frac{1}{2}} = 0,$$

or

$$\Big\{\{y-m(x+1)\}\ \{y-mx\}\Big\}^{\frac{1}{2}} + \Big\{\{y+m(x+1)\}\ \{y+mx\}\Big\}^{\frac{1}{2}}$$

$$+ 2m\{x(2x+1)\}^{\frac{1}{2}} = 0,$$

and this equation rationalized and reduced gives

$$(2x-1)(2x+1)y^2 = 4m^2x^3(2x+1)$$

which resolves into the two

$$2x+1=0$$

and $$y^2 = \frac{4m^2x^3}{2x-1}$$

the first of these, since

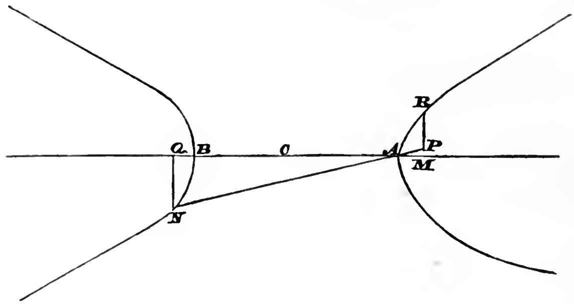
$$\nu = w(\mathrm{N}w - \mathrm{K}) = -\mathrm{K}x(2x+1)$$

requires $\nu = 0$, which would reduce the equation of the system to $\lambda v w + \mu u w = 0$, or $w = 0$, $\lambda v + \mu u = 0$, representing only two straight lines. Hence $2x+1=0$ must be rejected, and therefore $y^2 = \dfrac{4m^2x^3}{2x-1}$ is the required locus.

Now this curve is essentially one of the third order, and generated from the hyperbola in the same manner as the cissoid of Diocles is from the circle. This we proceed to demonstrate.

If x be measured in the contrary direction O A, the equation may be written

$$y^2 = \frac{4\, m^2\, x^3}{2\, x + 1}$$



Let a hyperbola be described of which the semi axes are, real $= \frac{1}{4}$, imag. $\frac{m\ \sqrt{2}}{4}$, and through the vertex A draw any line N A P. Draw N Q and make C M = C Q. Also draw the ordinate R M, cutting N A P in P, then P will be a point in the curve.

For let $y = a\, x$ be equation to N A P, A being origin, equation to hyperbola $y^2 = m^2\, x + 2\, m^2\, x^2$, $\therefore$ for intersection N, $a^2\, x = m^2 + 2\, m^2\, x$

$$x = -\frac{m^2}{2\, m^2 - a^2}; \quad \therefore \text{A Q} = \frac{m^2}{2\, m^2 - a^2}$$

Also B Q = A P = x, $\frac{m^2}{2\, m^2 - a^2} - x = \text{A B} = \frac{1}{2}$

$$\frac{2\, m^2}{2\, m^2 - a^2} = 2\, x + 1\, ;$$

$$\therefore \; 2\,m^2 - a^2 = \frac{2\,m^2}{2\,x + 1}$$

$$a^2 = \frac{4\,m^2\,x}{2\,x + 1} = \frac{y^2}{x^2}$$

$$\therefore \; y^2 = \frac{4\,m^2\,x^3}{2\,x + 1};$$

Hence the locus of P is the curve in question.

To find the asymptotes.

Taking the equation

$$y^2 = \frac{4\,m^2\,x^3}{2\,x - 1}$$

$$y = \pm\, 2\,m\,x^{\frac{3}{2}}\,(2\,x)^{-\frac{1}{2}} \left\{1 - \frac{1}{2\,x}\right\}^{-\frac{1}{2}}$$

$$= \pm\,\sqrt{2}\,m\,x \left\{1 + \frac{1}{4\,x} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{4\,x^2}\ \&c.\right\}$$

so that for x infinite we have

$$y = \pm\,\sqrt{2}\,m \left(x + \frac{1}{4}\right);$$

Moreover since $x = \frac{1}{2}$ makes y and $\frac{d\,y}{d\,x}$ infinite,

the equation $\qquad 2\,x - 1 = 0$

gives another asymptote.

These asymptotes being drawn, the curve will be found to consist of three distinct branches, as in the figure.

10°. What we have hitherto exhibited seems to be far from being the full extent of applicability of this method of investigation, as the following will show.

Let $A\, v\, w + B\, u\, w + C\, u\, v = 0 \ldots (3)$

be a *fixed* conic section passing through the intersections of $u = 0$, $v = 0$, $w = 0$. It is required to find a system of conic sections, each of which shall touch the lines $u = 0$, $v = 0$, $w = 0$, and also the curve (3).

Let $(\lambda\, u)^{\frac{1}{2}} + (\mu\, v)^{\frac{1}{2}} + (\nu\, w)^{\frac{1}{2}} = 0 \ldots (4)$

be the equation to any curve of the system.

This already touches u, v, w, and if we assume

$$(A\, \lambda)^{\frac{1}{2}} + (B\, \mu)^{\frac{1}{2}} + (C\, \nu)^{\frac{1}{2}} = 0 \ldots (\delta)$$

and investigate the envelope of (4) we find it to be no other than the equation (3).

Hence the equation (4), in which λ, μ, ν, are subject to the condition (δ), represents the required system.

Hence the locus of the centres of the system is

$$\{A\, L\, (L\, u - K)\}^{\frac{1}{2}} + \{B\, M\, (M\, v - K)\}^{\frac{1}{2}} + \{C\, N\, (N\, w - K)\}^{\frac{1}{2}} = 0.$$

11°. Now let $(A\, u)^{\frac{1}{2}} + (B\, v)^{\frac{1}{2}} + (C\, w)^{\frac{1}{2}} = 0 \ldots (5)$

be a fixed conic section touching $u = 0$, $v = 0$, $w = 0$,

and let it be required to find a system of conic sections, each passing through the mutual intersections of $u = 0$, $v = 0$, $w = 0$, and also touching (5).

Let $\lambda\, v\, w + \mu\, u\, w + \nu\, u\, v = 0 \ldots (6)$

be the equation to each curve of the system, and suppose λ, μ, ν connected by (δ) as before.

On investigating the envelope of (6) we find it to be no other than (5). Hence (6), subject to condition (δ), represents the system required.

The locus of the centres in this case will be

$$\{\mathrm{A}\, u\, (\mathrm{L}\, u - \mathrm{K})\}^{\frac{1}{3}} + \{\mathrm{B}\, v\, (\mathrm{M}\, v - \mathrm{K})\}^{\frac{1}{3}}$$
$$+ \{\mathrm{C}\, w (\mathrm{N}\, w - \mathrm{K})\}^{\frac{1}{3}} = 0.$$

This curve is double the dimensions of that in the preceding case, and each result assures us that were we to find the solution of the following, "To find the locus of the centres of systems of conic sections, each of which touches four given conic sections," we should have an algebraical curve of very high dimensions, and not in general resolvable into factors, each representing a curve of the second order.

I will conclude this chapter by applying my

method to solve a theorem proposed by Mr. Coombe in his Smith's Prize Paper of the present year.

The theorem is, "If a conic section be inscribed in a quadrilateral, the lines joining the points of contact of opposite sides, each pass through the intersection of the diagonals."

Let $u = 0$, $v = 0$, $w = 0$, $t = 0$, be the equations to the sides of the quadrilateral;

Then determining A, B, C, by making

$$\mathrm{A}\,u + \mathrm{B}\,v + \mathrm{C}\,w = t, \textit{ identically} \ldots (1)$$

And subjecting λ, μ, ν, to the condition

$$\frac{\lambda}{\mathrm{A}} + \frac{\mu}{\mathrm{B}} + \frac{\nu}{\mathrm{C}} = 0 \ldots (2)$$

we have $(\lambda\, u)^{\frac{1}{2}} + (\mu\, v)^{\frac{1}{2}} + (\nu\, w)^{\frac{1}{2}} = 0 \ldots (3)$

for the inscribed conic section.

But equation (3) may be put in the form

$$4\,\mu\,\nu\,v\,w = (\lambda\,u - \mu\,v - \nu\,w)^2$$

so that $\qquad \lambda\,u - \mu\,v - \nu\,w = 0$

is the equation to the line joining the points of contact of v and w.

From (1) we have

$$\mathrm{A}\,u + \mathrm{B}\,v \text{ identical with } t - \mathrm{C}\,w,$$

so that either of these equated to zero will represent the diagonal D B, and similarly

$A u + C w = 0$, or $t - B v = 0$ will represent the diagonal A C.

But from
$$A u + B v = 0$$
$$A u + C w = 0$$

and
$$\frac{\lambda}{A} + \frac{\mu}{B} + \frac{\nu}{C} = 0$$

Eliminating A, B, C, we obtain
$$\lambda u - \mu v - \nu w = 0$$

Hence this line passes through the intersection of the two diagonals. But this has been shown to be the line joining the points of contact of the opposite sides v, w; such line of contact therefore passes through the intersection of diagonals. Similarly, the other line of contact also passes through the intersection of diagonals.

CHAPTER III.

I HAVE applied analysis after the method followed in the last Chapter to the solution of a vast number of very general theorems, and always with complete success. I have also extended it to three dimensions, and have discovered many remarkable properties and relations hitherto unknown, and have also obtained very concise and elegant demonstrations of known theorems. It is intended in this Chapter to instance a few of them.

Let u, v, w, t be linear functions of x, y, z, and let the planes

$$u = 0, \quad v = 0, \quad w = 0,$$

be supposed to touch a surface of the second order in points situated in the plane $t = 0$; then the equation to that surface will be as follows:

$$A^2 u^2 + B^2 v^2 + C^2 w^2 - 2 AB uv - 2 AC uw - 2 BC vw \pm t^2 = 0.$$

For suppose $u = 0$, the equation becomes

$$(B v - C w)^2 \pm t^2 = 0.$$

Taking the upper sign this requires

$$B v - C w = 0, \text{ and } t = 0;$$

and taking the lower sign

$$B v - C w + t = 0, \text{ and } B v - C w - t = 0.$$

In either case the point determined by

$$u = 0, \quad B v - C w = 0, \text{ and } t = 0,$$

will be a point in the surface to which the plane $u = 0$ is tangential. In the former case it will touch the surface in this point only, in the latter it will touch in this point and cut in the straight lines

$$B v - C w + t = 0, \text{ and } B v - C w - t = 0.*$$

In the same way $v = 0$, and $w = 0$ are also tangent planes.

Now $B v - C w = 0$ represents a plane through the common intersection of the planes $v = 0$, $w = 0$. Similarly $A u - B v = 0$ represents a plane through the common intersection of $u = 0$, $v = 0$; and $C w - A u = 0$ one through the intersection of $w = 0$, $u = 0$.

Whence, since $C w - A u = 0$ is a consequence of $A u - B v = 0$, and $B v - C w = 0$, we may assert the following theorem:

If a surface of the second order be tangential to three planes, the planes passing through the

* This is the case of the hyperboloid of one sheet. If t be obliterated there is but one line, and the surface becomes a cone whose vertex is the common intersection of u, v, w.

mutual intersections of every two of them and the point of contact of the third tangent plane, will intersect in the same straight line.

Again, let straight lines be drawn from the point of mutual intersection of $u = 0$, $v = 0$, $w = 0$, one in each plane, and let two surfaces of the second order touch the three planes $u = 0$, $v = 0$, $w = 0$, in points respectively situated in those straight lines, then the equations to the two surfaces differing only in the value of t, we have at their points of intersection

$$t^2 - t'^2 = 0,$$

$$\text{or } t - t' = 0, \quad t + t' = 0.$$

Hence if two surfaces of the second order touch three planes in such a manner that the lines joining points of contact on each plane all pass through the point of common intersection of the three planes, the surfaces (if they intersect at all) intersect in one plane, or else in two planes.

In M. Chasles' Memoirs on Cones and Spherical Conics, translated by the Rev. Charles Graves, F.T.C.D., I find the following remark:

"These theorems might also be demonstrated by algebraic analysis; but this method, which in general offers so great advantages, loses them all

in this case, since it often requires very tedious calculations, and exhibits no connexion between the different propositions; so that it is only useful in verifying those which are already known, or whose truth has been otherwise suggested as probable."

All who have read M. Chasles' Memoirs must greatly admire the exquisite ingenuity and generalization displayed in them; but I think no one who well understands the use of analysis, and is capable of applying it to the utmost advantage, would readily subscribe to the preceding remark.

I would unhesitatingly engage to furnish good analytical demonstrations of all M. Chasles' theorems, and as the matter is allied to the subject of this volume, and would furnish perhaps a happy illustration to this part of it, I have adopted the suggestion of a scientific friend to devote this Chapter to the analytical investigation of some of Chasles' properties.

Let $u = 0$, $v = 0$ be two planes through the origin, $w = 0$ another plane, $r^2 = x^2 + y^2 + z^2$.

Then $$r^2 = m\,u\,v + n\,w \ldots (1),$$
represents a surface of the second order, of which

$u = 0$, $v = 0$, are cyclic planes, and $w = 0$ may be called the *metacyclic* plane.

For $u =$ const. reduces equation (1) to that of a sphere. The plane $u =$ const. necessarily intersects this sphere in a circle. But the surface and plane intersect in the same curve as the sphere and plane; therefore $u =$ const. intersects the surface in a circle. That is, all planes parallel to $u = 0$ intersect the surface in circles, or in other words $u = 0$ is a cyclic plane.

Similarly $v = 0$ is a cyclic plane.

If the origin be a point on the surface w is homogeneous in x, y, z, and is a tangent plane to the surface in the origin.

For $w = 0$ reduces (1) to $r^2 = m\,u\,v$, from which, on eliminating z, we obtain evidently a homogeneous linear equation in x and y, which will represent either the origin or two straight lines through the origin. In either case $w = 0$ is a tangent plane.*

* This is also evident from the consideration, that when the constant term in the general equation of the second or any higher order is zero, the *linear* part of the equation represents a plane touching the surface represented by the whole equation, in the origin.

Moreover u, v, and w are proportional to the perpendiculars from a point x, y, z, upon those planes respectively, and may be taken equal to such perpendiculars by the introduction of proper multipliers. Hence if ϕ, ϕ', θ are the angles which r makes with u, v, w, we have

$$\frac{u}{r} = \sin\phi, \quad \frac{v}{r} = \sin\phi', \quad \frac{w}{r} = \sin\theta,$$

by the substitution of which in (1) we obtain the condition

$$r\{1 - m\sin\phi\sin\phi'\} = n\sin\theta.$$

Hence the following theorem.

If in any surface of the second order a point be taken at which a tangent plane is drawn, and if, moreover, a chord whose length is r be drawn from the assumed point, and θ be the angle it makes with the tangent plane, ϕ and ϕ', the angles with the cyclic planes, then

$$r\{1 - m\sin\phi\sin\phi'\} = n\sin\theta,$$

where m and n are constant.

When $n = 0$, the surface becomes a cone, origin at vertex, and $\sin\phi\sin\phi' =$ const., which is one of Chasles' theorems.

When the chord is in either cyclic plane,

$$r = n\sin\theta.$$

Whence if δ be the diameter of one of the circular sections through the origin, and η the angle of inclination of its plane to the tangent plane,

$$\delta = n \sin \eta, \quad \therefore n = \frac{\delta}{\sin \eta}.$$

This therefore determines n, which will in general be different for different points on the surface. Also m will be the same for all points on the surface, because if the origin be changed, the axes remaining parallel to themselves, the terms of the second order remain the same.

In order therefore to determine m, suppose the origin changed to the extremity of one of the principal axes of the solid, the greatest or least (not the mean). Let A be the length of such axis, ε the angle it makes with either cyclic plane, D the diameter of a circular section through the extremity of such axis, then since

$$\sin \theta = 1, \text{ for } \theta = 90^\circ, \text{ and } n = \frac{D}{\cos \varepsilon},$$

we have

$$A \{1 - m \sin^2 \varepsilon\} = \frac{D}{\cos \varepsilon}; \quad \therefore m = \frac{A - D \sec \varepsilon}{A \sin^2 \varepsilon}.$$

The equation of the surface therefore becomes

$$r^2 = \frac{A - D \sec \varepsilon}{A \sin^2 \varepsilon} u v + \frac{\delta}{\sin \eta} w.$$

From this the reader will be able to deduce many singularly beautiful properties of great generality.

I will here instance one or two of them :

If two intersecting concentric surfaces of the second order have the same cyclic planes, they will intersect in a spherical conic or curve such that the product of the perpendiculars from any point in it on the cyclic planes is constant.

Let
$$r^2 = m\,u\,v + b^2,$$
$$r^2 = m'\,u\,v + b'^2,$$
be the two surfaces.

Eliminating $u\,v$, we have
$$(m' - m)\,r^2 = m'\,b^2 - m\,b'^2,$$
the equation to a sphere.

Also
$$(b'^2 - b^2)\,r^2 = (m\,b'^2 - m'\,b^2)\,u\,v,$$
the equation to a cone vertex at origin.

Also
$$(m - m')\,u\,v = b'^2 - b^2;$$
$$\therefore \quad u\,v = \frac{b'^2 - b^2}{m - m'} = \text{const.},$$
but u, v are the perpendiculars from a point x, y, z on the cyclic planes; this product is therefore constant for all points on the common intersection of the two surfaces.

Again let Δ be the diameter of a sphere touching a surface of the second order in two points, *and intersecting that surface in circular sections,* then $\Delta = \dfrac{\delta}{\sin \eta}$, and the equation of the surface is

$$r^2 = \frac{A - D \sec \varepsilon}{A \sin^2 \varepsilon} u v + \Delta w,$$

the origin being in the surface, and $w = 0$ a tangent plane through the origin.

Let $w = z$, and make $z = 0$, then

$$\mathrm{r}^2 = \frac{A - D \sec \varepsilon}{A \sin^2 \varepsilon} \mathrm{u}\, \mathrm{v},$$

r, u, v being the values of r, u, v when $z = 0$.

This is a homogeneous equation of the second order in x and y, and will therefore represent either the point of contact, or one or two straight lines. When it represents one straight line the surface is conical; when two it is the hyperboloid of one sheet, the two straight lines being generatrices. Hence the product of the sines of the angles made by either generatrix with the cyclic planes is constant and equal to $\left(\frac{\mathrm{u}}{\mathrm{r}} \cdot \frac{\mathrm{v}}{\mathrm{r}}\right) = \dfrac{A \sin^2 \varepsilon}{A \sim D \sec \varepsilon}$.

We now proceed to a few of the properties of Cones.

Generation of Cones of the second degree, and their Supplementary Cones.

General principle.

Let

$$A x + B y + C z = 0, \quad \frac{x}{A} = \frac{y}{B} = \frac{z}{C},$$

be a moveable plane and straight line perpendicular thereto. If the condition to which the motion of the plane or line be subjected be such that combined or not with the equation

$$A^2 + B^2 + C^2 = 1$$

it leads to a homogeneous equation of the second order in A, B, C, then the plane will envelope a cone of the second degree, and the line will generate another cone of the second degree supplementary to the former.

For let $$f(A, B, C) = 0,$$

be the homogeneous relation above supposed. Then since x, y, z, in the equations to the straight line are proportional to A, B, C, we can replace the latter by the former in the *homogeneous* equation, and thus have $f(x, y, z) = 0$, for the surface described by the moveable straight line.

Now let $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0,$$

be the equation of the cone so described by the

straight line $$\frac{x}{A} = \frac{y}{B} = \frac{z}{C};$$

Then $$\frac{A^2}{a^2} + \frac{B^2}{b^2} - \frac{C^2}{c^2} = 0.$$

To find the surface enveloped by

$$A x + B y + C z = 0,$$

we have $$\frac{A}{a^2} d A + \frac{B}{b^2} d B - \frac{C}{c^2} d C = 0,$$

$$x d A + y d B + z d C = 0;$$

$$\therefore \quad \frac{A}{a^2} = \lambda x, \quad \frac{B}{b^2} = \lambda y, \quad - \frac{C}{c^2} = \lambda z,$$

or $A = \lambda a^2 x, \quad B = \lambda b^2 y, \quad C = - \lambda c^2 z.$

Putting these in

$$A x + B y + C z = 0,$$
$$a^2 x^2 + b^2 y^2 - c^2 z^2 = 0,$$

the equation to the other cone.

This, therefore, establishes the general principle, and consequently when any property is predicated of such cones, the only thing necessary to be done to demonstrate it is to find whether combined or not with the condition

$$A^2 + B^2 + C^2 = 1,$$

it leads to a homogeneous result of the second order in A, B, C.

For example. "The sum or the difference of the angles which each focal line makes with a side of the cone of the second degree is constant;" or in other words, if a straight line drawn from the point of intersection of two given straight lines makes angles with them whose sum or difference is constant, the moveable line traces the surface of a cone.

Let the equations of the given lines be

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c},$$

$$\frac{x}{a'} = \frac{y}{b'} = \frac{z}{c'},$$

and of the moveable line

$$\frac{x}{\mathrm{A}} = \frac{y}{\mathrm{B}} = \frac{z}{\mathrm{C}},$$

θ and θ' the two angles;

$$\therefore \quad \theta \pm \theta' = 2\,\alpha \text{ const.};$$

$$\therefore \quad \cos(\theta \pm \theta') = \cos 2\,\alpha,$$

$$\text{or } \cos^2\theta + \cos^2\theta' - 2\cos 2\,\alpha \cos\theta \cos\theta' = \sin^2 2\,\alpha;$$

$$\text{but} \quad \cos\theta = \mathrm{A}\,a \pm \mathrm{B}\,b + \mathrm{C}\,c,$$

$$\cos\theta' = \mathrm{A}\,a' + \mathrm{B}\,b' + \mathrm{C}\,c'.$$

Putting these in the above, and multiplying $\sin^2 2\alpha$ by $A^2 + B^2 + C^2$ ($= 1$), we have

$$(A a + B b + C c)^2 + (A a' + B b' + C c')^2$$
$$- 2 \cos 2\alpha (A a + B b + C c) (A a' + B b' + C c')$$
$$= \sin^2 2\alpha (A^2 + B^2 + C^2),$$

a homogeneous relation of the second order, and therefore the proposition is true.

But besides establishing the truth of the proposition, we are enabled immediately to find the equation of the cone so traced, thus:

$x, y, z,$ being written for A, B, C in the above relation, making

$$u = a x + b y + c z,$$
$$u' = a' x + b' y + c' z,$$
$$r^2 = x^2 + y^2 + z^2,$$

we have for the equation in question

$$r^2 \sin^2 2\alpha = u^2 + u'^2 - 2 u u' \cos 2\alpha.$$

The lines perpendicular to the planes $u = 0$, $u' = 0$, are called focal lines for the following reason. Consider u' const. and $= p'$, then p' is the perpendicular from the origin on the plane $u' - p' = 0$, and $\sqrt{r^2 - p'^2}$ is therefore the distance from a point x, y, z, in the section of the cone made by

$u' - p' = 0$ and the point in which p' intersects that plane.

The above equation gives

$$(r^2 - p'^2) \sin^2 2\alpha = (u - p' \cos 2\alpha)^2;$$

$$\therefore \sqrt{r^2 - p'^2} = \textit{linear} \text{ function of } x, y, z.$$

This is a property of the focus and the focus only. Hence p' passes through the foci of all sections perpendicular to it. Similarly, a line perpendicular to $u = 0$ passes through foci of all sections parallel to this plane.

Again, if the moveable line makes angles with the fixed line, such that the product of their cosines is constant, it will trace a cone of the second order. The notation being as before, we have

$$\cos\theta \cos\theta' = \text{const.} = n,$$

$$\text{or } (\mathrm{A}a + \mathrm{B}b + \mathrm{C}c)(\mathrm{A}a' + \mathrm{B}b' + \mathrm{C}c') = n(\mathrm{A}^2 + \mathrm{B}^2 + \mathrm{C}^2),$$

a homogeneous equation of the second order.

This establishes the proposition and gives for the equation of the cone $n r^2 = u u'$, wherein $u = 0$, $u' = 0$, are called cyclic planes for the following reason.

Consider u' constant and $= p'$, then

$$r^2 = \frac{1}{n} \cdot p' u,$$

which being the equation to a sphere, the sections parallel to $u' = 0$, will be circular.

Similarly, sections parallel to $u = 0$ are circular.

If two cones be supplementary to each other, the cyclic planes of the one will be perpendicular to the focal lines of the other.

Let the two supplementary cones be denoted by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0,$$

$$a^2 x^2 + b^2 y^2 - c^2 z^2 = 0,$$

in which a is supposed greater than b.

Eliminate x^2 from the first by means of the equation $r^2 = x^2 + y^2 + z^2$, and it becomes

$$r^2 = a^2 \left\{ \left(\frac{1}{a^2} + \frac{1}{c^2} \right) z^2 - \left(\frac{1}{b^2} - \frac{1}{a^2} \right) y^2 \right\}.$$

Hence the two cyclic planes are

$$\left(\frac{a^2 + c^2}{c^2} \right)^{\frac{1}{2}} z \pm \left(\frac{a^2 - b^2}{b^2} \right)^{\frac{1}{2}} y = 0,$$

$$\text{or} \quad u = \frac{b}{a}\left\{\frac{a^2 + c^2}{b^2 + c^2}\right\}^{\frac{1}{2}} z + \frac{c}{a}\left\{\frac{a^2 - b^2}{b^2 + c^2}\right\}^{\frac{1}{2}} y = 0,$$

$$u' = \frac{b}{a}\left\{\frac{a^2 + c^2}{b^2 + c^2}\right\}^{\frac{1}{2}} z - \frac{c}{a}\left\{\frac{a^2 - b^2}{b^2 + c^2}\right\}^{\frac{1}{2}} y = 0.$$

Also eliminating x^2 from the other cone,

$$r^2 = \frac{1}{a^2}\{(a^2 + c^2) z^2 + (a^2 - b^2) y^2\},$$

but if $\cos 2\alpha = \dfrac{b^2 - c^2}{b^2 + c^2}$, it is easily found that this last equation may be put in the form

$$r^2 \sin^2 2\alpha = u^2 + u'^2 - 2uu' \cos 2\alpha.$$

But we have shown that this is the form when the perpendiculars to $u = 0$, $u' = 0$, are focal lines. Hence the cyclic planes $u = 0$, $u' = 0$, of the first cone are at right angles to the focal lines of the second or supplementary cone.

Now, *incidentally*, we have also proved that 2α the sum or difference of angles which any side of the second cone makes with its focal lines is independent of *a*. Hence, if a system of cones has the same vertex and axis, and be such that sections made by a plane perpendicular to the axis, all have the same major axis, the sum or difference of the angles which any side of one

of the cones of the system makes with its focal lines will be constant, not merely for the same cone, but also for all the cones of the system, this sum or difference being $= 2 \tan^{-1} \frac{c}{b}$.

No one will fail to notice here the striking analogy between spherical and plane conics.

Again, let the plane of $x\ y$ be parallel to one system of circular sections, and let the vertex of the cone be the origin, and the line through the centres of the circles be the axis of z, which line will in general be inclined to plane $x\ y$.

Take the axes of x, y perpendicular to each other.

The equation of the cone will be

$$x^2 + y^2 = n^2 z^2.$$

Now taking z constant and $n z = a$, suppose we have $x^2 + y^2 = a^2$, and whatever property be proved in plano respecting this circle, there will necessarily be a corresponding one of the cone. There will therefore be no greater analytical difficulty in proving the conical property than that in plano.

For example. "If two tangent planes be drawn to a cone of the second order such that their traces

on a cyclic plane are always inclined at the same angle, the intersection of such planes will trace out another cone of the second order having a cyclic plane in common with the first cone."

The analytical proof will be as follows.

The equations to the two tangent planes are

$$x \cos(\theta + \alpha) + y \sin(\theta + \alpha) = n z,$$
$$x \cos(\theta - \alpha) + y \sin(\theta - \alpha) = n z.$$

Adding and dividing by $2 \cos \alpha$,

$$x \cos\theta + y \sin\theta = n z \sec \alpha.$$

Subtracting $\quad x \sin\theta - y \cos\theta = 0.$

Taking sum of squares $x^2 + y^2 = n^2 z^2 \sec^2 \alpha$, which is another cone having $x\,y$ for a cyclic plane.

As another illustration take the following.

Let there be two given straight lines which intersect, and let a plane perpendicular to the line bisecting the angle between them be drawn, then if two planes revolve about the given lines such that their traces on the transversal plane include a constant angle, the intersections of such planes will trace out a cone of the second order which shall have one of its cyclic planes parallel to the transversal plane.

This proposition is in fact tantamount to proving

that if the base and vertical angle of a triangle be constant, the locus of the vertex is a circle, and it is from this plane proposition that Chasles infers the conical one.

The following is the analysis.

The equations to the revolving planes have the form

$$(x + m z) \cos (\theta + \alpha) + y \sin (\theta + \alpha) = 0,$$
$$(x - m z) \cos (\theta - \alpha) + y \sin (\theta - \alpha) = 0.$$

The elimination of θ immediately gives

$$(x^2 + y^2) \sin 2 \alpha = m^2 z^2 \sin 2 \alpha - 2 m z y \cos 2 \alpha,$$

which is a cone of the second order having circular sections parallel to $x\,y$.

It surely cannot be said that analysis loses any of its usual advantages in the cases here adduced. For my own part I always conclude that when analysis does seem to lose any of its usual advantages, the fault is not in the analysis, but in the want of dexterity and clearness of analytical conception in the analyst.

I am now about to make a remark to which I think considerable importance is to be attached.

Whatever a plane problem may be, we may always consider it as the result of one or more

relations between two variables or unknown quantities x and y.

Put $\frac{x}{z}$ for x and $\frac{y}{z}$ for y, and we are sure to have the corresponding conical problem.

Thus in the several investigations of Chapter II. if we conceive throughout $\frac{x}{z}$ and $\frac{y}{z}$ to be put for x and y, and moreover consider z constant *until the final result is obtained*, we shall have conical properties corresponding to each of the plane properties.

It will be sufficient to enunciate one or two, as the reader will easily supply the rest.

If a system of cones touch the four planes of a tetrahedral angle, the diameters of the several individuals of that system conjugate to a given fixed plane, will all lie in the same plane.

If a system of cones pass through the four edges of a tetrahedral angle, the diameters of the several individuals of that system conjugate to a given fixed plane, will trace out another cone of the second order.

If a system of cones pass through two of the edges of a pentahedral angle and touch the two

opposite sides, the diameters of the several individuals of that system conjugate to a given fixed plane will trace out another cone of the second order.

I have shown, therefore, how from any plane problem a conical one may be deduced, and to this class of conical problems I would propose the name of plano-conical problems. There is an equally extensive class arising from the intersection of cones and concentric spheres, to which the term sphero-conical problems might with propriety be applied. These requiring a different management the following illustrations are supplied.

"If through two fixed intersecting right lines two rectangular planes be made to revolve, their intersection will trace out a cone of the second order passing through the fixed right lines and having its cyclic planes at right angles to them.

This is another of Chasles' theorems.

Let $\frac{x}{\mathrm{A}} = \frac{y}{\mathrm{B}} = \frac{z}{\mathrm{C}}$ be the equations of the generating line.

Let the fixed lines be in the plane $x\ z$ inclined

at an angle a to axis of z. By the property of right-angled spherical triangles, we have $\cos 2a$ equal product of cosines of generating line with fixed lines,

or $(C \cos a + A \sin a)(C \cos a - A \sin a) = \cos 2a$, where $A^2 + B^2 + C^2 = 1$.

This equation is therefore

$$A^2 \cos^2 a + B^2 \cos 2a - C^2 \sin^2 a = 0,$$

$$\text{or} \quad x^2 \cos^2 a + y^2 \cos 2a - z^2 \sin^2 a = 0.$$

This is the equation to a cone, and on making $y = 0$, it reduces to

$$x^2 \cos^2 a - z^2 \sin^2 a = 0,$$

$$\text{or} \quad x \cos a - z \sin a = 0, \quad x \cos a + z \sin a = 0,$$

and, therefore, the surface passes through the two fixed lines of which these are the equations.

Eliminating y by the equation $r^2 = x^2 + y^2 + z^2$,

$$x^2 \cos^2 a + (r^2 - x^2 - y^2) \cos 2a - z^2 \sin^2 a = 0,$$

$$\text{or} \quad r^2 \cos 2a = z^2 \cos^2 a - x^2 \sin^2 a,$$

and therefore the cyclic planes are

$$z \cos a + x \sin a = 0,$$

$$z \cos a - x \sin a = 0,$$

which are perpendicular to the given lines.

2. Let a system of cones of the second order pass through the four edges of a tetrahedral angle,

to find the surface traced by the axes of each individual of the system, or in other words, required the locus of the spherical centres of a system of spherical conics each passing through four fixed points on the sphere.

Let u and v be two homogeneous functions of x, y, z of second order, so that $u = 0$, $v = 0$ may represent two cones of that order. Suppose them to intersect in four lines, then $u + \lambda v = 0$ will for different values of λ represent all the cones having the same vertex, and passing through the same lines, for any point in any of the lines makes $u = 0$, $v = 0$ separately, and therefore satisfies the above equation.

Now in order to find the equations for the directions of the axes we have, first considering z as a function of x and y,

$$\frac{du}{dx} + \frac{du}{dz} \cdot \frac{dz}{dx} + \lambda \left\{ \frac{dv}{dx} + \frac{dv}{dz} \cdot \frac{dz}{dx} \right\} = 0,$$

$$\frac{du}{dy} + \frac{du}{dz} \cdot \frac{dz}{dy} + \lambda \left\{ \frac{dv}{dy} + \frac{dv}{dz} \cdot \frac{dz}{dy} \right\} = 0.$$

Putting $\frac{dz}{dx} = -\frac{x}{z}$, $\frac{dz}{dy} = -\frac{y}{z}$, conditions which insure the perpendicularity of the straight line

represented by the preceding with its conjugate plane, and thus making it peculiar to the axes, and then eliminating λ, the resulting equation is

$$x\left\{\frac{du}{dy}\cdot\frac{dv}{dz}-\frac{du}{dz}\cdot\frac{dv}{dy}\right\}+y\left\{\frac{du}{dz}\cdot\frac{dv}{dx}-\frac{du}{dx}\cdot\frac{dv}{dz}\right\}$$

$$+z\left\{\frac{du}{dx}\cdot\frac{dv}{dy}-\frac{du}{dy}\cdot\frac{dv}{dx}\right\}=0.$$

Now u, v being homogeneous and of the second order, $\frac{du}{dx}$, &c. will be homogeneous and of the first order. The preceding will therefore be homogeneous and in general of the third order. Hence, classifying the curves described on a spherical surface by the orders of the equations of the concentric cones by whose intersection with the spherical surface they are produced, it will follow that the locus of the centres of a system of spherical conics of the second order passing through four given points will be a spherical conic in general of the third order.

POSTSCRIPT.

It has seemed necessary to the Author, though he is not a Member of the Senate of the Cambridge University, to say a few words with reference to a work which has lately appeared by Dr. Whewell, entitled "Of a liberal Education in general, and with particular reference to the leading Studies of the University of Cambridge."

What has been advanced in the preceding pages is addressed chiefly to professed mathematicians, and is intended to express the humble opinion of the Author as to the supremacy of Analytical Mathematics. It is by no means questioned or denied that appropriate ideas both in Geometry, and other subjects, which are most successfully carried on analytically, ought in the first instance to be attained, and this by the study of such works as Dr. Whewell classifies under the title of "permanent studies." If the ideas appropriate to each particular branch be not first distinctly engraved on the mind, it is to little purpose that we have recourse to the aid of analysis; but having once possessed ourselves of those ideas, analysis becomes a powerful instrument for combining and generalizing to an extent which may well be called infinite. The mode, however, of applying

it is a great matter. In the hands of a person of little skill it often leads to a labyrinth of perplexities and false conclusions. Dr. Whewell, after asserting at p. 43 that "Analytical reasoning is no sufficient discipline of the reason, on account of the way in which it puts out of sight the subject matter of the reasoning," further adds, that "The analyst does not retain in his mind, in virtue of his peculiar processes, *any apprehension* of the differences of the things about which he is supposed to be reasoning."

In answer to the first statement, it seems sufficient to say that "analytical reasoning" cannot be charged with putting out of sight the subject matter of the reasoning on which it is employed, any more than the pen, ink, and paper of the Author can be charged with this concealment, for the former is just as much an instrument as the latter. If Bœvius strikes the lyre from which Horace drew such sweet and lively strains, is the lyre in fault because it does not reproduce them? And in reference to the other remark, if an analyst "does not retain in his mind any apprehension of the differences of the things about which he is supposed to be reasoning," he is plainly a person who does not know what he is about, and can only be *supposed* to be reasoning by those who know not how he ought to reason.

So strange does the whole of Dr. Whewell's reasoning, in order to prove that "Analysis is of little value as a discipline of the reason for general purposes," appear to us that we strongly suspect he has not had in view that which mathematicians understand by the term

"analysis," but that his object has been to "counteract, correct, and eradicate" a vicious system of mathematical education in the University, and which he ascribes to the use of symbols. The Doctor is perfectly right as to the existence of this vicious system, and very right, as one of the guardians of education, to endeavour to correct it, but most decidedly do we deny that *analysis* is to be blamed for it. The fact is, that for several years past it has been the custom for incipient graduates, after having passed the Senate House Examination with more or less credit, to take pupils. Now those tutors (many of them highly estimable men, and men of sterling talent,) are often very inexperienced, probably most of them thoroughly ignorant of mathematics only three years previous to entering on their tutorial occupations. But education, like every thing else, requires study, thought, and experience. A young tutor may be a highly talented individual, and yet a very bad teacher. He has forgotten the difficulties experienced by himself, and perhaps never known those experienced by others. He cannot believe it possible that any being born with reasoning faculties can stumble in going over the Pons Asinorum, or fail to understand the Binomial Theorem, and a thousand other minutiæ. He therefore takes too much for granted as to the state of knowledge of his pupil, and all pupils are anxious to conceal rather than display ignorance.

The tutor, therefore, taking it for granted that his pupil is already possessed of the appropriate preliminary ideas, ushers him too rapidly into the domains of ana-

lysis. The pupil appears to progress rapidly, feels perfectly satisfied both with himself and his tutor, and soon begins to fancy that he *may be* Senior Wrangler. He can solve any quadratic, separate roots, draw tangents and asymptotes, differentiate and integrate like harlequin, and all this after having read only three terms. But he cannot do the simplest deduction from Euclid, has no idea of a geometrical limit, makes sad bungling of a statical problem, and does not understand Taylor's Theorem.

This state of things continuing, he is ushered into Dynamics, Lunar and Planetary Theories, &c. &c. and becomes ready for the Tripos, from which he emerges last of the Senior Ops, retires from Cambridge, and is puzzled all his life long to find out what is the use of a University education.

Had such a youth been in the hands of an experienced person and distinguished mathematician, who would have taken care to *educate* him properly, who would have carefully *completed the links* connecting geometrical and algebraical reasoning, who would in every analytical investigation have kept the ideas appropriate to the subject of investigation constantly before the mind of the pupil, who would have shown him in many cases the identity of analytical and geometrical reasoning, and that in all cases the former is as it were the sublimation of the latter; then indeed it would not have been

"Parturiunt montes, nascitur ridiculus mus."

The preceding pages are intended to show to the mathematicians of this country what may be done even

on a very common subject in the way of further developement and generalization, by one, who while he is employing analysis with all the skill of which he is capable, never loses sight of appropriate ideas, but has throughout the whole investigation those ideas vividly in his mind. Had the work been principally intended for learners, explanation would of course have been more copious.

R. M. Coll. Sandhurst,
23rd Oct. 1846.

William Stevens, Printer, Bell Yard, Temple Bar.

5